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THE REPRESENTATION OF A CLASS OF TWO STATE STATIONARY PROCESSES IN TERMS OF INDEPENDENT RANDOM VARIABLES

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Introduction. Consider a stationary random process $\{X_n; n=0,\pm 1,\ldots\}$. It has recently been of some interest to find conditions on $\{X_n\}$ such that one can construct a process $\{Y_n\}$ with the same probability structure as $\{X_n\}$ on the space of independent identically distributed random variables $\{\xi_n\}$ (say uniformly distributed on $[0,1]$) by a Borel function $f(\xi_0,\xi_{-1},\ldots) = f(\xi)$ $(\xi = (\ldots,\xi_{-1},\xi_0,\xi_1,\ldots))$ and its shifts, that is,

$$Y_n = f(T^n\xi),\ n=0,\pm 1,\ldots \quad (1)$$

Here f is to be a function of $\xi_0,\xi_{-1},\ldots$ and T is the shift operator. It can be shown that a necessary condition is that $\{X_n\}$ be purely nondeterministic or regular (in the terminology of A. N. Kolmogorov)[5]. In fact it has been shown that the necessary and sufficient condition for such a representation in the case of a countable state Markov chain is that it be regular [6]. Most of the sufficient conditions for such a representation in the case of a continuous state Markov process contain something reminiscent of a Doeblin condition [2]. We explicitly mention and state an interesting sufficient condition of this type obtained recently by Hanson:

(i) $\{X_n\}$ is a stationary regular Markov process

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(ii) There exist Borel sets A,B of the state space and a nonnegative measure ϕ on the state space such that $\bar{P}(B) > 0$ ($\bar{P}$ is the stationary measure induced by the process on the state space), $\phi(A) > 0$ and for all $x \in B$ and $A' \subset A$ the transition measure

$$P(x,A') \geq \phi(A'). \tag{2}$$

It is clear that it would be extremely interesting to investigate a class of processes which would not satisfy a Doeblin-like condition even when they are imbedded naturally in a Markov process. The two state stationary regular processes are such a class. Moreover these processes are of considerable interest in their own right, particularly in communication problems.

Much of this paper will concern itself with a reinterpretation of results of Doeblin and Fortet [1] and Harris [3] with respect to such a representation.

Preliminaries. Since we will imbed the two state processes in Markov processes in a natural way, it will be reasonable to give a sufficient condition for a representation of type (1) for Markov processes. Let $\{Y_n; n=0,\pm 1,\ldots\}$ be a stationary Markov process. We can just as well assume the moment $E|Y_n|$ exists since a nonlinear instantaneous transformation can effect this. Further assume there is a sequence of independent random variables ξ_n uniformly distributed on [0,1] with $\xi_n, \xi_{n+1}, \ldots$ independent of $Y_n, Y_{n-1}, \ldots$ and a family of transformations $T_\xi(\cdot)$(with $T_\xi(y)$

a Borel function of (ξ, y)) <u>such that</u> $T_{\xi_n}(Y_n) = Y_{n+1}$ <u>with probability one</u>. One can then prove the following theorem just as in [5].

<u>Theorem 1.</u> <u>There is a representation of type</u> (1) <u>for</u> $\{Y_n\}$ <u>in terms of the</u> $\{\xi_n\}$ <u>process if and only if there is a set</u> S <u>of</u> y_1 <u>points of measure one such that</u>

$$T_{\xi_n} T_{\xi_{n-1}} \cdots T_{\xi_1} y_1 - T_{\xi_n} T_{\xi_{n-1}} \cdots T_{\xi_1} y_1' \longrightarrow 0 \tag{3}$$

<u>in</u> $\xi_1, \ldots, \xi_n, \ldots$ <u>measure for all</u> $y_1, y_1' \in S$ <u>as</u> $n \longrightarrow \infty$.

Our object is to study stationary processes with two states, say 0 and 1. We shall now imbed such a process in a Markov process following Harris. If we let $Y_n = (X_n, X_{n-1}, \ldots)$ the new process $\{Y_n; n=0,\pm 1, \ldots\}$ is a stationary Markov process where the k^{th} coordinate of the Y_n vector $\underset{k}{(Y_n)} = X_{n-k}$. The points y are halfway infinite sequences of zeros and ones. The Borel field is generated by sets of the form $\underset{k}{(y)} = x_k$. The transition function is essentially given by

$$\phi(y) = P[X_{n+1} = 1 | (X_n, X_{n-1}, \ldots) = y] \tag{4}$$

which can be taken as a Borel function in y. If we set

$$z = \frac{x_0}{2} + \frac{x_1}{2^2} + \ldots \tag{5}$$

a real number z, $0 \leq z \leq 1$, is obtained. All real numbers z have a unique binary representation except for those which have a finite binary expansion. These have two binary expansions, either with a final infinite sequence of zeros ...1000... or a final infinite sequence of ones ...0111... . However, the set of these numbers is countable. All points of this type must have measure zero because of the stationarity of the measure except possibly for .000... or .111... . We can therefore adopt Harris' convention in which the binary expansion of 1 is taken as .111... and in all other ambiguous cases the expansion terminating in zeros is preferred to that terminating in 1's. The random vectors Y_n will be identified with the real numbers given by the binary expansion (5) subject to the convention adopted above. Thus the Markov process $\{Y_n\}$ is to be regarded as one with state space the real numbers y, $0 \leq y \leq 1$, and transition mechanism

$$1-P[Y_{n+1} = \tfrac{1}{2} y | Y_n = y] = \phi(y) = P[Y_{n+1} = \tfrac{1}{2} + \tfrac{1}{2} y | Y_n = y]. \quad (6)$$

A regularity condition on the transition function $\phi(y)$ due to Doeblin and Fortet [ ] is now described. We use the notation $(x\equiv y)_m$ to mean that the first m digits in the binary expansion of x are the same as those in y. Let

$$\varepsilon_m = \sup_{(x\equiv y)_m} |\phi(x)-\phi(y)|. \quad (7)$$

Note that the ε_m are a monotone sequence of numbers. The condition introduced by Doeblin and Fortet is

$$\sum_m \varepsilon_m < \infty. \quad (8)$$

We shall assume either

$$\phi(y) \geq \Delta > 0 \quad 0 \leq x \leq 1 \quad (9)$$

or

$$1-\phi(y) \geq \Delta > 0 \quad 0 \leq x \leq 1.$$

Notice that (9) implies that $\varepsilon_0 < 1$.

<u>Some Results on Convergence.</u> We shall first give in some detail a theorem due to Harris. It is given in detail because Harris just briefly gave a sketch of the proof and it does play a basic role in the development of one of our results. The method is related to an idea of Doeblin.

Let $\xi_1, \xi_2, \ldots$ be independent random variables uniformly distributed on $[0,1]$. Define the processes $Y_n = Y_n(y)$, $Y_n' = Y_n(y')$, $0 \leq y, y' \leq 1$, as follows. Let $Y_1 = y$, $Y_1' = y'$ and given Y_n, Y_n'

$$Y_{n+1} = \begin{cases} \frac{1}{2} Y_n & \text{if} \quad \xi_n > \phi(Y_n) \\ \frac{1}{2} + \frac{1}{2} Y_n & \text{if} \quad \xi_n \leq \phi(Y_n) \end{cases} \tag{10}$$

with the corresponding representation of Y_{n+1}' in terms of Y_n' and ξ_n.

<u>Theorem 2</u>. <u>If conditions</u> (8) <u>and</u> (9) <u>are satisfied then</u>

$$P(|Y_n - Y_n'| > \varepsilon) \longrightarrow 0 \tag{11}$$

<u>for each fixed</u> $\varepsilon > 0$ <u>as</u> $n \longrightarrow \infty$ <u>uniformly in</u> y, y'.

Let $U_n = 1$ $(U_n' = 1)$ if $Y_{n+1} = (1+Y_n)/2$ (if $Y_{n+1}' = (1+Y_n')/2$).

Then

$$P(U_n \neq U_n' | Y_n, Y_n') \leq |\phi(Y_n) - \phi(Y_n')|. \tag{12}$$

We first want to show that almost every pair of sequences $\{Y_n\}$, $\{Y_n'\}$ (both generated by the ξ_n's) have a k-tuple of U coincidences $(U_m=U_m', \ldots, U_{m+k-1}=U_{m+k-1}')$ with probability one. This follows by contradiction in the following manner. Suppose this is not the case. Then there is a set A (of ξ sequences) of positive probability with no k-tuple of U coincidences. For large enough n there is a set A_n, $A \subset A_n$, with $P(A_n) < (1+\delta)P(A)$ $(\delta > 0)$ that is measurable with respect to the Borel field generated by $\xi_1, \ldots, \xi_n$. Let

$$B_{k,n} = \{U_n=U_n', \ldots, U_{n+k-1}=U_{n+k-1}'\} \; . \tag{13}$$

Then

$$\begin{aligned} &P(A_n \cap B_{k,n}) \\ &= \int_{A_n} P\{B_{k,n} \mid Y_n, Y_n'\} \, dP \\ &\geq (1-\varepsilon_0)(1-\varepsilon_1)\cdots(1-\varepsilon_k)P(A_n) . \end{aligned} \tag{14}$$

But then

$$\begin{aligned} P(A \cap B_{k,n}) &= P(A_n \cap B_{k,n}) - P((A_n - A) \cap B_k) \\ &\geq (1-\varepsilon_0)\cdots(1-\varepsilon_k)P(A_n) - \delta P(A) > 0 \end{aligned} \tag{15}$$

if δ is sufficiently small leading to a contradiction. Of course, this argument is valid for every finite k.

Let C_m be the set of ξ sequences for which the first k-tuple of U coincidences occurs (and starts) at m. Then the C_m are disjoint and $\sum_{m=1}^{\infty} P(C_m) = 1$ by the argument of the previous paragraph. Given any fixed $\delta > 0$, for sufficiently large n

$$\sum_{m=1}^{n} P(C_m) > 1-\delta \ . \tag{16}$$

Note that we will have $|Y_{n+\ell}-Y'_{n+\ell}| < \varepsilon$ if ℓ is such that $2^{-\ell} < \varepsilon$ and we have ℓ successive U coincidences $U_n = U'_n, \ldots, U_{n+\ell-1} = U'_{n+\ell-1}$. Thus if $\ell > k$

$$\begin{aligned} &P(|Y_{n+\ell}-Y'_{n+\ell}| < \varepsilon) \\ &\geq \sum_{m=1}^{n} P(C_m \cap \{U_{m+k} = U'_{m+k}, \ldots, U_{n+\ell} = U'_{n+\ell}\}) \\ &\geq \prod_{j=k}^{\infty} (1-\varepsilon_j) \sum_{m=1}^{n} P(C_m) \geq (1-\delta) \prod_{j=k}^{\infty} (1-\varepsilon_j) \end{aligned} \tag{17}$$

by inequality (15). If k is sufficiently large and δ small enough this will be close to one since $\Sigma\varepsilon_j < \infty$. Thus given any $\varepsilon > 0$, $P(|Y_n-Y'_n| > \varepsilon) \longrightarrow 0$ as $n \rightarrow \infty$ uniformly in y, y'.

The following corollary was of primary interest to Harris.

Corollary. Under conditions (8) and (9)

$$F_n(z|y) = P(Y_n \leq z|Y_1 = y) \longrightarrow G(z) \tag{18}$$

uniformly in y as $n \rightarrow \infty$. $G(z)$ is the one and only stationary distribution for the Y_n process.

For a given y one can find a sequence n_1 such that

$$F_{n_1}(z|y) \longrightarrow G(z|y) \tag{19}$$

weakly. From (11) it follows that $F_{n_1}(z|y') \longrightarrow G(z|y)$ weakly independently of y' and hence $G(z|y) = G(z)$ is independent of y.

Further the convergence is strict at continuity points z of $G(z)$. Given any $\varepsilon > 0$ for a given continuity point z there is a value of n sufficiently large so that for $n_1 > n$

$$|F_{n_1}(z|y) - G(z)| < \varepsilon \tag{20}$$

independently of y. Then

$$F_n(z|y) = \int F_{n_1}(z|y')dF_{n-n_1}(y'|y) \tag{21}$$

and hence $F_n(z|y) \longrightarrow G(z)$ uniformly in y as $n \longrightarrow \infty$. If $\phi(1) = 1(\phi(0)=0)G(z)$ must have a jump of magnitude 1 at 1(0). If this is not the case one can show the limit distribution G is continuous. Let $\phi_0(y) = 1-\phi(y)$, $\phi_1(y) = \phi(y)$. It is clear that

$$F_{n+1}(z|y) = \sum_{j=0}^{1} \int_0^{2z-j} \phi_j(y')dF_n(y'|y). \tag{22}$$

Because of the regularity condition (8) we can go to the limit and obtain

$$G(z) = \sum_{j=0}^{1} \int_0^{2z-j} \phi_j(y)dG(y), \tag{23}$$

an equation that any stationary distribution for the Y_n process must satisfy. The uniqueness of the stationary distribution G is obvious.

We also mention an interesting result due to Karlin [4].

<u>Theorem 3</u> (Karlin). <u>If the function</u> ϕ <u>is monotone with</u>

$$|\phi(x)-\phi(y)| \leq \mu < 1 \tag{24}$$

<u>then</u>

$$F_n(z|y) = P(Y_n \leq z|Y_1=y) \to G(z)$$

where G is independent of y as $n \to \infty$.

Almost as an immediate result of Karlin's theorem we have the following result

Theorem 4. If the function ϕ is monotone with (24) satisfied then

$$P(|Y_n - Y_n'| > \varepsilon) \to 0$$

for each $\varepsilon > 0$ as $n \to \infty$ for all y, y'.

We shall assume ϕ monotone nondecreasing since the case of ϕ monotone nonincreasing can be reduced to this by interchanging 0's and 1's in the original X_n process. Further let us take $\phi(0) \neq 0$, $\phi(1) \neq 1$ since otherwise we are in the trivial situation in which $G(z)$ has a jump of magnitude 1 at 0 or 1. We have already remarked that then G must be continuous. Consider the transformations $T_\xi(y)$, $0 \leq \xi \leq 1$, as given by (10), that is

$$T_\xi(y) = \begin{cases} \frac{1}{2}y & \text{if } \xi > \phi(y) \\ \frac{1}{2}+\frac{1}{2}y & \text{if } \xi \leq \phi(y). \end{cases} \tag{25}$$

All the functions $T_\xi(\cdot)$ are monotone nondecreasing since $\phi(y)$ is monotone nondecreasing. Hence, given any y, y' with $0 \leq y < y' \leq 1$

$$T_\xi(y) \leq T_\xi(y') \tag{26}$$

for all ξ, $0 \leq \xi \leq 1$. Thus if $Y_n = T_{\xi_{n-1}} \cdots T_{\xi_1} y$,

$Y_n' = T_{\xi_{n-1}} \cdots T_{\xi_1} y'$ with $y \le y'$ it follows that $Y_n' \ge Y_n$. However, we know that Y_n, Y_n' have a common limiting distribution $G(z)$ which is continuous . But

$$P\{Y_n' > z, Y_n \le z\} + P\{Y_n' \le z\} = P\{Y_n \le z\} \tag{27}$$

so that

$$P\{Y_n' > z, Y_n \le z\} \to 0 \tag{28}$$

at every point z as $n \to \infty$. Given any $\varepsilon > 0$ one can find $k(\varepsilon)$ points $z_1 < z_2 < \ldots < z_k$ such that

$$\bigcup_{j=1}^{k} \{Y_n \le z_j, Y_n' > z_j\} \supset \{Y_n' - Y_n > \varepsilon\}.$$

It therefore follows that $P\{Y_n' - Y_n > \varepsilon\} \to 0$ for every $\varepsilon > 0$ as $n \to \infty$ and hence $Y_n' - Y_n \to 0$ in probability as $n \to \infty$. This has been carried out for any two points y, y' with $y < y'$. However, given any two points y, y' we can find a third point y'' (assuming both $y, y' > 0$) such that $y'' < y, y'$ and therefore $Y_n' - Y_n \to 0$ as $n \to \infty$ for all y, y'.

<u>Representation of 0-1 Processes</u>. We now briefly state and prove the main result.

<u>Theorems</u>. <u>Let</u> $\{Y_n; n=0, \pm 1, \ldots\}$ <u>be a stationary two state process</u> (states 0 and 1) <u>with transition function</u> $\phi(y)$ <u>satisfying either</u> (8) <u>and</u> (9) <u>or monotone with</u> (24). <u>There is then a sequence of independent random variables</u> $\{\xi_n\}$ <u>uniformly distributed on</u> $[0,1]$ <u>such that</u> $\{Y_n\}$ <u>has the one-sided representation</u> (1) <u>in terms of the</u> ξ_n.

It is clear that there is a stationary process satisfying either conditions (8) and (9) or (24) by the argument of the corollary in either of these cases. Further the probability structure of such a stationary process is uniquely determined by the transition function $\phi(y)$. Let $\{Y_n; n=0,\pm1,\ldots\}$ be such a stationary process. Given the $\{Y_n\}$ process let $\{\eta_n; n=0,\pm1,\ldots\}$ be a sequence of independent random variables uniformly distributed on $[0,1]$ and independent of the $\{Y_n\}$ process. We shall now construct a sequence of independent random variables $\{\xi_n\}$ uniformly distributed on $[0,1]$ that satisfy the assumptions of Theorem 1. Of course the $\{Y_n\}$ sequence and $\{\xi_n\}$ sequence will be dependent. Let δ be the function on 0,1 such that $\delta(0) = 0$ and $\delta(1) = 1$. Set

$$\begin{aligned} \xi_n &= \delta(2Y_{n+1}-Y_n)\eta_n\phi(Y_n) \\ &\quad + (1-\delta(2Y_{n+1}-Y_n))\{\phi(Y_n)+\eta_n(1-\phi(Y_n))\} . \end{aligned} \tag{30}$$

Now

$$\begin{aligned} P\{\xi_n \le t | Y_n\} &= P\{\eta_n \le t/\phi(Y_n) | Y_n\}\phi(Y_n) \\ &\quad + P\{\eta_n \le [t-\phi(Y_n)]/[1-\phi(Y_n)] | Y_n\}(1-\phi(Y_n)). \end{aligned} \tag{31}$$

If $0 \le t \le \phi(Y_n)$ the second term on the right of (31) is zero and we have

$$P\{\xi_n \le t | Y_n\} = t, \quad 0 \le t \le \phi(Y_n). \tag{32}$$

If $\phi(Y_n) < t \le 1$ expression (31) becomes

$$\phi(Y_n) + \frac{t-\phi(Y_n)}{1-\phi(Y_n)}(1-\phi(Y_n)) \tag{33}$$

so that

$$P\{\xi_n \leq t | Y_n\} = t \tag{34}$$

for all t, $0 \leq t \leq 1$. Further

$$P\{\xi_n \leq t_n, \ldots, \xi_1 \leq t_1 | Y_1\} \tag{35}$$

$$= \int_{\xi_{n-1} \leq t_{n-1}} \cdots \int_{\xi_1 \leq t_1} P\{\xi_n \leq t_n | \xi_{n-1}, \ldots, \xi_1, Y_1\} P\{\xi_{n-1} | \xi_{n-2}, \ldots, \xi_1, Y_1\} \cdots P\{\xi_1 | Y_1\}$$

$$= \int_{\xi_{n-1} \leq t_{n-1}} \cdots \int_{\xi_1 \leq t_1} P\{\xi_n \leq t_n | Y_n\} P\{\xi_{n-1} | Y_{n-1}\} \cdots P\{\xi_1 | Y_1\}$$

$$= t_n t_{n-1} \cdots t_1 \text{ , } \quad 0 \leq t_n, \ldots, t_1 \leq 1.$$

Clearly the random variables $\{\xi_n\}$ are such that $\xi_n, \xi_{n+1}, \ldots$ are independent of $Y_n, Y_{n-1}, \ldots$ and the family of transformations $T_\xi(\cdot)$ as given by (25) are such that $T_{\xi_n}(Y_n) = Y_{n+1}$ with probability one. Under conditions (8) and (9) it follows by Theorems 2 and 1 that the $\{Y_n\}$ process has a representation of type (1) in terms of the ξ sequence. Under monotonicity and condition (24) Theorems 3 and 1 imply that the $\{Y_n\}$ process has a representation of type (1) in terms of the ξ sequence.

<u>An Interpretation</u>. An amusing interpretation of some of these results can be given in the case of Markov chains. Suppose there are two stations, the first transmitting messages to the second. We shall assume that the message that the first station

wishes to transmit to the second has two states 0 and 1 and is Markov of finite order with stationary transition mechanism. However, there is a possibility that a third party may also receive the message. This is to be avoided and for that reason station one decides to encode the message so that the encoded message appears to be a realization of a sequence of independent identically distributed random variables. Of course, the proper decoder has to be supplied to station two. Assuming no distortion in transmission, there will still be some loss of information in the decoding and we shall try to get some measure of this loss of information if the encoding and decoding mechanism are set up in a manner consistent with the discussion given earlier.

We first consider the case in which the data which station one wishes to transmit to the second station is a two state Markov process (say X_n) with transition probability matrix

$$\begin{pmatrix} 1-p & p \\ 1-q & q \end{pmatrix} \qquad \begin{matrix} 0 < p,q < 1 \\ p \neq q \end{matrix} \tag{36}$$

The message that station one wishes to transmit need not be stationary or have an infinite past. One can just formally agree that the first Y_1 is any number between 0 and 1 whose first entry in its binary expansion is X_1. Clearly in terms of (36) the ϕ function will be given by

$$\phi(y) = P\{Y_{n+1} = \tfrac{1}{2} + \tfrac{1}{2}y \mid Y_n = y\} = \begin{cases} p & \text{if } 0 \leq y < \frac{1}{2} \\ q & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases} \tag{37}$$

and the Y_n's are generated recursively from Y_1 by the random mechanism specified by (37). Station one will then encode the sequence $\{Y_n\}$ in the $\{\xi_n\}$ sequence as given by

$$\xi_n = \begin{cases} \eta_n p & \text{if } Y_{n+1} = \frac{1}{2} + \frac{1}{2} Y_n & 0 \leq Y_n < \frac{1}{2} \\ \eta_n q & \text{if } Y_{n+1} = \frac{1}{2} + \frac{1}{2} Y_n & \frac{1}{2} \leq Y_n \leq 1 \\ p+\eta_n(1-p) & \text{if } Y_n = \frac{1}{2} Y_n & 0 \leq Y_n < \frac{1}{2} \\ q+\eta_n(1-q) & \text{if } Y_n = \frac{1}{2} Y_n & \frac{1}{2} \leq Y_n \leq 1 \end{cases} \tag{38}$$

where the η_n's are independent random variables uniformly distributed on [0,1] and independent of the $\{Y_n\}$ sequence. Now let us consider the appropriate decoding procedure to be applied to the $\{\xi_n\}$ sequence received by station two in order to recover as much as possible of the original $\{X_n; n=1,2,\ldots\}$ sequence that station one transmitted in encoded form. Suppose $p < q$. If $\xi_n < p$ it is clear that $X_{n+1} = 1$ while if $\xi_n > q$ then $X_{n+1} = 0$. If $p < \xi_n < q$ $(X_n, X_{n+1}) = (0,1)$ or $(1,0)$ and this indeterminary remains until the first ξ_k is encountered undershooting p or overshooting q. Once such a ξ_k is encountered X_{k+1} is determined and all the following X_n's, $n > k+1$, are determined by the following ξ_n values, $n \geq k+1$. However, the previous values X_n, $n \leq k$, can not be recovered. Once ξ_k is observed and X_{k+1}

determined, the following X_n's are determined recursively by setting

$$X_{n+1} = \begin{cases} 1 & \text{if } X_n=0,\ 0 \leq \xi_n < p \\ & \text{or } X_n=1,\ 0 \leq \xi_n < q \\ 0 & \text{if } X_n=0,\ p < \xi_n \leq 1 \\ & \text{or } X_n=1,\ q < \xi_n \leq 1. \end{cases} \tag{39}$$

The random observations ξ_n that station two receives are, of course, independent and identically distributed on [0,1]. Thus, the expected time for the first ξ_n (that is, ξ_k) undershooting p or overshooting q to be observed is given by

$$E(k) = \Sigma k\alpha^{k-1}(1-\alpha) = (1-\alpha)^{-1} \tag{40}$$

with $\alpha = q-p$. If α is not close to one this mean time will be moderate in size.

A more general case of interest is that in which station one wishes to transmit a two state k^{th} order Markov chain $\{X_n\}$ with stationary transition mechanism, that is, the ϕ function depends on y only through the first k places in the binary representation of y. For convenience, assume that the range of ϕ is bounded away from zero and one. Then

$$\phi(y) = p(x_1,\ldots,x_k) \quad \text{if} \quad \sum_{j=1}^{k} \frac{x_j}{2^j} \leq y < \sum_{j=1}^{k} \frac{x_j}{2^j} + \frac{1}{2^k}\,, \tag{41}$$

$x_i = 0,1$ and

$$\xi_n = \begin{cases} \eta_n p(x_1,\ldots,x_k) & \text{if } Y_{n+1} = \frac{1}{2} + \frac{1}{2} Y_n \\ & \text{and } \sum_{j=1}^{k} \frac{x_j}{2^j} \le Y_n < \sum_{j=1}^{k} \frac{x_j}{2^j} + \frac{1}{2^k} \\ p(x_1,\ldots,x_k)+\eta_n(1-p(x_1,\ldots,x_k)) & \text{if } Y_{n+1} = \frac{1}{2} Y_n \\ & \text{and } \sum_{j=1}^{k} \frac{x_j}{2^j} \le Y_n < \sum_{j=1}^{k} \frac{x_j}{2^j} + \frac{1}{2^k} \end{cases} \tag{42}$$

where the $\{\eta_n\}$ are independent random variables uniformly distributed on $[0,1]$ and independent of the $\{Y_n\}$ (or equivalently $\{X_n\}$) sequence. Let

$$p = \min_{x_i} p(x_1,\ldots,x_k) \tag{43}$$

$$q = \max_{x_i} p(x_1,\ldots,x_k).$$

Again the ξ_n's transmitted by station one will be independent and uniformly distributed on $[0,1]$. As before if $\xi_n < p$ then $X_{n+1} = 1$ while if $\xi_n > q$ X_{n+1} must be zero. Once one determines the values of k neighboring X's, say $X_n, X_{n+1}, \ldots, X_{n+k-1}$, all following X's can be determined by these and $\xi_{n+k-1}, \xi_{n+k}, \ldots$. $X_n, X_{n+1}, \ldots, X_{n+k-1}$ will be determined if none of the values $\xi_{n-1}, \ldots, \xi_{n+k-2}$ fall in the range p to q. Let j be the first time such a run of ξ values $\xi_{j-k+1}, \ldots, \xi_j$ falling outside the range p to q arise. As before let $\alpha = q-p$. Then the mean time for the first such run of $k\xi$ values to occur is

$$E(j) = \alpha\{(1-\alpha)^{-k} - 1\} . \tag{44}$$

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